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Internal Geometry of Hadron Resonances

S. Bruce[1] and P. Salgado[1]

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Motivated by previous work on high-energy quantum mechanics, a simple model is devised to study the internal geometry of hadron resonances. In this model we assume new basic canonical commutation relations between the (internal) coordinate and momentum operators of the hadronic quantum system. By systematically imposing Lie algebra commutation relations between these and other observables, we discuss the free and bound particle problems, identifying in each case the corresponding internal symmetries. For the bound particle problem, which models quark confinement, this symmetry turns out to be characterized by Dirac's two-oscillator representation of the $O(3, 2)$ de Sitter group.

Some years ago Saavedra and Utreras (1981; Montesinos *et al.*, 1985; Talukdar and Niyagi, 1982; Giffon and Predazzi, 1983) considered the possible advantage of new kinematics in the description of the internal dynamics of relativistic particles. Specifically, they proposed a high-energy generalization of the usual canonical commutation relations (crr) of quantum mechanics (QM) with the purpose of reinterpreting some results of quark physics. They pointed out that the ccr between the canonical coordinates (q, p) giving rise to Heisenberg's uncertainty principle were abstracted from atomic physics, whose characteristic energies are of the order of a few electron volts. This fact then raises the question (Saavedra and Utreras, 1981; Saavedra, 1981) of whether they will still be valid at high energies in the range of 10^9–10^{12} eV.

Saavedra and Utreras (1981) were concerned basically with a 1-dimensional problem. In this paper we present a simple model for treating the corresponding 3-dimensional problem. See also Montesinos *et al.* (1985) for a nonalgebraic approach to the 3d problem.

If such a model is conceivable, the new basic commutation relations between the coordinate and momentum operators are required to be

[1]Department of Physics, University of Concepción, Casilla 3-C, Concepción, Chile.

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constants of motion (Saavedra and Utreras, 1981; Saavedra, 1981). In three dimensions this is achieved by demanding that the canonical coordinates (q_i, p_j) satisfy

$$[q_i, p_j] = if(O_1, \ldots, O_n)\delta_{ij}, \qquad i, j = 1, 2, 3$$
$$[H, f(O_1, \ldots, O_n)] = 0 \tag{1}$$

where f is a certain function of the n observables of the system. Equation (1) should reduce to the usual ccr in the appropriate nonrelativistic limit. In addition to this, we will consider a physical system where q_i, p_j, and $f(O_a)$ close under a Lie group algebra forming a dynamical symmetry of the system. Here we depart from Saavedra and Utreras (1981), who did not impose this condition to any extent. In fact, in their treatment of the harmonic oscillator problem, the corresponding commutation relations (cr) do not close under a Lie algebra.

To be quite general, one should not consider (q, p) to be just ordinary phase space canonical coordinates. They could also represent generalized internal coordinates of a physical system. The Zitterbewegung of the Dirac electron readily presents an example of this possibility (Saavedra, 1965, 1981; Barut and Bracken, 1981*a*,*b*; Barut and Thacker, 1985).

To develop the model, we shall make the simple choice

$$f(O_\alpha) = f(H) = rI + tH + \frac{s}{2} H^2 \tag{2}$$

with r, t, s real constants, and H the Hamiltonian of the system. A direct generalization of the 1d theory would imply setting $s = 0$ with $t \neq 0$, leading to the correct description of 3d Zitterbewegung. However, it will not give, in this approach, the desired answer when applied to (high-energy) quark physics. In what follows we shall see that here it is more appropriate to set $t = 0$ and $s \neq 0$.

We notice that in three dimensions equations (1) and (2) introduce further restrictions on the remaining cr. To see this, we shall assume (Saavedra and Utreras, 1981) the validity of the Heisenberg equation of motion, so that the q_i and the p_j are related through

$$[K, q_i] = i\hbar sc^2\left(\frac{1}{2c^2}\{H, \dot{q}_i\}\right) =: -i\hbar sc^2 p_i \tag{3}$$

where $K =: Ir + sH^2$. This relation gives a definition for the (relativistic) momentum operator[2] in terms of the space coordinates q_i.

[2]Notice that $p_i = (1/2c^2)\{H, \dot{q}_i\}$ is the usual expression for the canonical momentum of the unbound particle in relativistic quantum mechanics.

To study the free particle dynamical symmetry, we now proceed to compute the Jacobi identity for the triple (q_i, p_j, K). Using equations (1)–(3), we find

$$[p_i, p_j] = 0 \tag{4}$$

which says that the p_j continue to be compatible observables. However, computing the Jacobi identity for the triple (q_i, q_j, p_j), $i \neq j$, we obtain

$$[p_j, [q_i, q_j]] = (\hbar c)^2 s p_k, \qquad i, j, k \text{ cyclic} \tag{5}$$

Equation (5) shows that $[q_i, q_j] \neq 0$. This is an interesting result, since equations (1) and (2) induce a modification in the cr among the q_i. These commutators naturally define three new dynamical variables for the system. Let us write

$$[q_i, q_j] =: ig\epsilon_{ijk} J_k \tag{6}$$

where g is a real constant. Using equation (6) to calculate the Jacobi identity for the triple (K, q_i, q_j), we find that

$$[K, J_k] = 0 \tag{7}$$

Now it is easily checked, by calculating the various Jacobi identities among the $n = 10$ operators in the set $A = \{q_i, p_j, K, J_k\}$, that the Lie algebra containing these operators is given by

$$\begin{array}{lll} [q_i, p_j] = i\hbar K\delta_{ij}, & [J_i, J_j] = i\hbar\epsilon_{ijk} J_k, & [J_i, q_j] = i\hbar\epsilon_{ijk} q_k \\ [K, q_i] = -i\hbar s c^2 p_i, & [q_i, q_j] = -i\hbar c^2 s\epsilon_{ijk} J_k, & [J_i, p_j] = i\hbar\epsilon_{ijk} p_k \\ [K, J_k] = 0, & [K, p_k] = 0, & [p_i, p_j] = 0 \end{array} \tag{8}$$

where we have normalized J_k by $-\hbar s c^2/g$ to fit an angular momentum operator and thus g is also gauged away. In (8) we can distinguish three different cases: For $s < 0$, we have the Lie algebra of the Euclidean group $E(4)$. If $s = 0$, it reduces to the Lie algebra of the Heisenberg group (notice $J_k \equiv 0$ here). Finally, for $s > 0$, we obtain the Lie algebra of the Poincaré group. For this last case we must recall that the irreducible unitary representations of the algebra classify particles with unbounded momenta (Bargmann and Wigner, 1946).

In each of these alternatives, if we diagonalize the representing operator of K on an irreducible unitary representation space for the group, its eigenvalues will be real and continuous.

Next we want to study, within our approach, the bound particle problem. To this end we look for an appropriate value for the commutator between H^2 and p_i. The choice has to be consistent with the Lie algebra cr for the set A. It is easy to see that the only possible choice that keeps invariant the nonvanishing commutators in (8) is to set

$$[K, p_i] = \frac{s}{2}[H^2, p_i] =: i\hbar k^2 s q_i \tag{9}$$

with k being a real constant. We can further justify this statement by observing that (9) modifies the cr involving p_i in (8) (see below) leading to $so(4, 1)$ or $so(3, 2)$ de Sitter Lie algebras for A, with $s \neq 0$. These two Lie algebras are the only ones that under contraction ($k \to 0$) give the Poincaré Lie algebra (8). Note also that the same cr (9) is obtained for the Lie algebra describing the usual ($s = 0$) relativistic case involving a (Lorentz) scalar linear potential in a two-body problem (Ram, 1982).[3]

As mentioned before, equation (9) imposes further conditions on the cr of the set of observables A. With the help of the Jacobi identity and the antisymmetry of the Lie commutators we find that this set satisfies the cr

$$\begin{aligned}
&[q_i, p_j] = i\hbar K\delta_{ij}, && [J_i, J_j] = i\hbar\epsilon_{ijk}J_k, && [J_i, q_j] = i\hbar\epsilon_{ijk}q_k \\
&[K, q_i] = -i\hbar s c^2 p_i, && [q_i, q_j] = -i\hbar c^2 s\epsilon_{ijk}J_k, && [J_i, p_j] = i\hbar\epsilon_{ijk}p_k \\
&[K, p_i] = i\hbar s k^2 q_i, && [p_i, p_j] = -i\hbar k^2 s\epsilon_{ijk}J_k, && [K, J_k] = 0
\end{aligned} \tag{10}$$

We observe that A now closes on a ten-dimensional semisimple Lie algebra. It can be directly checked for $s < 0$, $s = 0$, and $s > 0$, (10) reduces to the Lie algebra of $SO(5)$, the oscillator group $Os(3)$, and the $SO(3, 2)$ de Sitter group, respectively.

Now we discuss the remarkable case $s > 0$. It is known (Evans, 1967) that there exist irreducible unitary representations of $SO(3, 2)$ which possesses positive-definite or negative-definite eigenvalues of K. These representations contract to the physical representations of the Poincaré group and they belong to the discrete series for $SO(3, 2)$. Among this set we single out the interesting ones given by Dirac (1963), for which K has the eigenvalues

$$r + \frac{s}{2}E_{(j)}^2 = \hbar c k s\left(j + \frac{1}{2}\right), \qquad j = 0, 1, 2, \ldots \tag{11a}$$

$$r + \frac{s}{2}E_{(j)}^2 = \hbar c k s\left(j + \frac{1}{2}\right), \qquad 2j = 1, 2, 3, \ldots \tag{11b}$$

here j is the angular momentum quantum number, with $\mathbf{J}^2 = \hbar^2 j(j + 1)$.

[3] In the limit $s \to 0$ the squared Hamiltonian becomes $H^2 = c^2p^2 + (mc^2 + kr)^2$ with $p_i = -i\hbar\partial/\partial x_i$. Note also that its spectrum reduces, for $l = 0$, to $E_n^2 = k(2n + 1)$, which is known to be independent of m.

From (10) we obtain the uncertainty relations

$$\begin{aligned}\Delta q_i \Delta p_j &\geq \frac{1}{2}\hbar|\langle K\rangle|\delta_{ij}\\ \Delta q_i \Delta q_j &\geq \frac{1}{2}\hbar c^2|s|\epsilon_{ijk}|\langle J_k\rangle|\\ \Delta p_i \Delta p_j &\geq \frac{1}{2}\hbar k^2|s|\epsilon_{ijk}|\langle J_k\rangle|\end{aligned} \tag{12}$$

In the ultrarelativistic limit the bound states will feel only the confining part of the potential. In this limit the square masses of hadrons, with given strangeness, isospin, etc., are found to be proportional to their angular momenta through the relation $E^2 = \beta j + \gamma$, where β is the Regge slope (Lucha *et al.*, 1991). The spectrum (11) properly fits this relation. Choosing $r = 1$, the value of s will be determined experimentally through β and γ. For example, for mesons with vanishing strangeness (Particle Data Group, 1988) $\beta = 2\hbar ck \cong 1.14\ \mathrm{GeV}^2$ we find that $s = +3.7\ \mathrm{GeV}^{-2}$.

In the model presented above resonances are described by the Dirac (infinite-dimensional) representation of the dynamical group $SO(3, 2)$ and thus they can be taken to be two-dimensional harmonic oscillators (Dirac, 1963). Due to its simplicity, this model seems to be more suited for describing, to a good approximation, some meson resonances since these quantum systems are naturally described as bound states of two constituent quarks.

As a final remark, we should mention that the Dirac representation of $SO(3, 2)$ has already been applied, although in a different context, to the study of space-time symmetries of relativistic particles (Han *et al.*, 1990; Kim and Kim, 1991; and references quoted therein). Han *et al.* (1990) and Kim and Kim (1991) exploit the geometry of the Wigner phase-space picture of the Dirac representation, which allows them to study the symmetry of the associated Lie algebra in terms of canonical transformations. In their picture, however, the generator K is related rather to the magnitude of the Pauli–Lubanski vector than to the magnitude of the squared Hamiltonian.

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